Exercise 3

(a) Find the slope of the tangent line to the parabola $y = 4x - x^2$ at the point $(1, 3)$

(i) using Definition 1 (ii) using Equation 2

(b) Find an equation of the tangent line in part (a).

(c) Graph the parabola and the tangent line. As a check on your work, zoom in toward the point $(1, 3)$ until the parabola and the tangent line are indistinguishable.

Solution

Definition 1 and Equation 2 give two ways of calculating the slope of a tangent line.

$$\text{Definition 1:} \quad m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

$$\text{Equation 2:} \quad m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Part (a)

The function in this exercise is $f(x) = 4x - x^2$, and the value of x that we want to know the tangent line at is $a = 1$. Find m with Definition 1.

$$\begin{aligned} m = \lim_{x \to 1} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1} \frac{(4x - x^2) - [4(1) - (1)^2]}{x - 1} &= \lim_{x \to 1} \frac{4x - x^2 - 3}{x - 1} \\ &= -\lim_{x \to 1} \frac{x^2 - 4x + 3}{x - 1} \\ &= -\lim_{x \to 1} \frac{(x-3)(x-1)}{x - 1} \\ &= -\lim_{x \to 1} (x - 3) \\ &= -(1 - 3) \\ &= 2 \end{aligned}$$

$$\begin{aligned} m = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} &= \lim_{h \to 0} \frac{[4(1+h) - (1+h)^2] - [4(1) - (1)^2]}{h} \\ &= \lim_{h \to 0} \frac{(4 + 4h) - (1 + 2h + h^2) - (4 - 1)}{h} \\ &= \lim_{h \to 0} \frac{2h - h^2}{h} \\ &= \lim_{h \to 0} (2 - h) = 2 - 0 = 2 \end{aligned}$$

Part (b)

The aim is to find the equation of the line with slope 2 that passes through $(1, 3)$. Start with the general formula of a line.

$$y = mx + b$$

Since the slope is 2, $m = 2$.

$$y = 2x + b$$

Use the fact that the line passes through $(1, 3)$ to determine b.

$$3 = 2(1) + b$$

$$3 = 2 + b$$

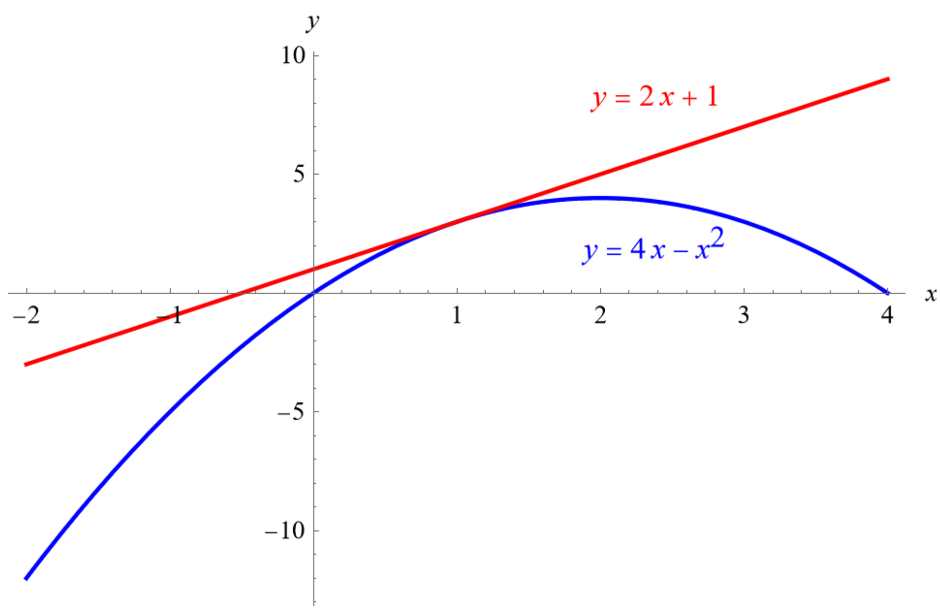
$$b = 1$$

Therefore,

$$y = 2x + 1.$$

Part (c)

Below is a plot of the parabola with the tangent line at $x = 1$.



Zoom in to the interval $0.9 \le x \le 1.1$.